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Bethe Lattice Spin Glass: The Effects of a Ferromagnetic Bias and External Fields. II. Magnetized Spin-Glass Phase and the de Almeida–Thouless Line

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In this and the companion paper, we analyze the $\pm J$ Ising spin-glass model on the Bethe lattice with fixed uncorrelated boundary conditions. Phase diagrams are derived as a function of temperature vs. concentration of ferromagnetic bonds and, for a symmetric distribution of bonds, external field vs. temperature. In this part we characterize magnetized spin-glass (MSG) phases by divergence of an appropriate susceptibility: at zero field this signals the existence of an intermediate MSG phase; at nonzero field, this is used to identify the de Almeida–Thouless line.

KEY WORDS: Spin glass; Bethe lattice; multicritical point.

1. INTRODUCTION

In this paper we continue our analysis of the $\pm J$ Ising spin-glass model on the Bethe lattice with fixed uncorrelated boundary conditions. This analysis began in the companion paper[1] with a discussion of the recursion relation for the distribution of single-site magnetizations $\rho(X)$. Certain aspects of the temperature vs. concentration phase diagram (Fig. 1) were derived rigorously using moment analysis and bifurcation theory. We found that at high temperatures the system is paramagnetic P, i.e., $\rho(X) = \delta(X)$. The spin-glass transition (P $\rightarrow$ SG) corresponds to an instability associated with

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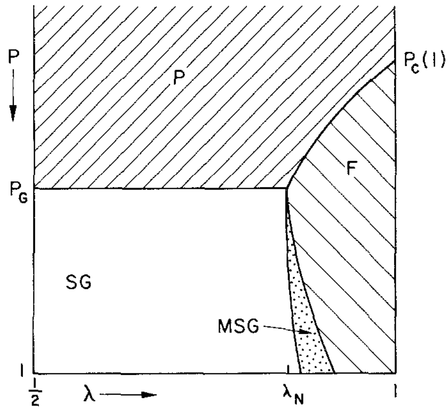

Fig. 1. Phase diagram for the Bethe lattice spin glass, plotted as a function of $p = \tanh(J/kT)$ and the fraction λ of ferromagnetic bonds. At high temperatures, the system is paramagnetic. As the temperature decreases, there is a transition to edither a spin-glass or a ferromagnetic phase, depending on λ. Between these phases there is an intermediate magnetized spin-glass (MSG). Like the ferromagnet, the MSG phase has nonzero net magnetization, but it also has glassy susceptibilities. The phase diagram for $\lambda < 1/2$ can be obtained by reflection across the line $\lambda = 1/2$, replacing F and MSG with phases which have long-range antiferromagnetic order. In this paper we determine the phase boundary between the magnetized spin-glass phase and the ferromagnetic phase. All other phase boundaries were determined in ref. 1.

the Edwards–Anderson order parameter q, which is the width of the distribution ρ. The ferromagnetic transition ($\mathrm{P} \to \mathrm{F}$) corresponds to an instability associated with the magnetization m, which is the mean of ρ. The transition from the spin-glass phase to a magnetized phase, which is this paper we will show is in fact a magnetized spin-glass phase ($\mathrm{SG} \to \mathrm{MSG}$), corresponds to an instability of the symmetric spin glass solution ρ_{SG} to perturbations associated with the mean m. However in the neighborhood of the multicritical point, the last phase boundary, $\mathrm{F} \to \mathrm{MSG}$, which corresponds to the instability of the ferromagnetic solution to glassy order, cannot be obtained using the methods employed to determine the other phase boundaries. In Section 2 we demonstrate the existence of this intermediate phase by calculating the Edwards–Anderson susceptibility χ_{EA}, which diverges at the glassy phase boundaries. At nonzero field, divergence of χ_{EA} also characterizes the de Almeida–Thouless line[2] (Fig. 2), as shown in Section 3.

In Section 4 we conclude this paper with a summary of the results obtained in this and the companion paper, and a discussion of the relationship between the Bethe lattice spin glass and the infinite-range model. We show that in the formal limit where the coordination number of the lattice tends to infinity, the recursion relation becomes the so-called SK

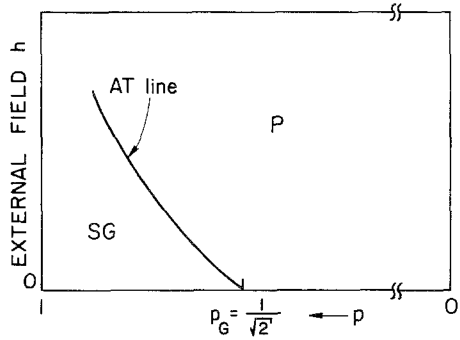

Fig. 2. De Almeida–Thouless line. On the Bethe lattice the spin-glass transition persists in the presence of an external field. The asymptotic form of the AT line is $h(T) \sim |p - p_G|^{3/2}$, and the critical exponent for the Edwards–Anderson susceptibility is $\gamma = 1$.

equation.[3] The solution of this equation is the replica-symmetric solution of the SK model. Thus, at least in a formal sense, the correct solution on the Bethe lattice is analogous to the (unphysical) replica-symmetric solution of the infinite-range model.

Finally, we note that the method used to calculate the Edwards–Anderson susceptibility also leads to a rather novel solution (on a different lattice) which may provide a clue to the relationship between the "replica-symmetric" solution on the Bethe lattice and the "replica-symmetry-breaking" solution of the infinite-range model. In these calculations we will consider a lattice constructed from two identical Bethe lattices joined ferromagnetically ar each site. A system of three coupled recursion relations, derived in Appendix A of the companion paper, describes this new lattice. Interestingly, we find that in addition of the single-lattice solution which we analyze in this and ref. 1, new solutions (which do not exist for a single lattice) emerge in the spin-glass phase. One of these appears to be closely related to the replica-symmetry-breaking solution of the infinite-range model.[4] Further analysis of the coupled latticed system may lead to a more complete understanding of the relationship between the two models.

2. EDWARDS–ANDERSON SUSCEPTIBILITY AND THE FERROMAGNET–MSG PHASE BOUNDARY

In this section we calculate the Edwards–Anderson susceptibility χ_{EA}:

$$\chi_{EA} = \frac{1}{N^2} \sum_{i,j} \overline{\langle \sigma_i \sigma_j \rangle^2 - \langle \sigma_i \rangle^2 \langle \sigma_j \rangle^2} \tag{1}$$

χ_{EA} is a nonlinear susceptibility, and it diverges approaching the spin-glass phase boundaries. It is worth noting that χ_{EA} is not exactly the same as the experimentally measured susceptibility, although both susceptibilities are quadratic. We except that higher even-order analogues of χ_{EA} should also diverge along the spin-glass boundary. We find that in the simplest case, χ_{EA} diverges crossing the spin-glass—paramagnet phase boundary. This result is obtained as a special case (zero field) of the calculation performed in Section 3, where we analyze the spin-glass transition in an external field. In this section, we find that near the multicritical point, χ_{EA} diverges in the magnetized phase before we reach the zero-magnetization phase boundary discussed in Section 6 of the companion paper. This calculation demonstrates the existence of a magnetized spin-glass phase in the neighborhood of the multicritical point, as illustrated in the phase diagram (Fig. 1). The zero-temperature results of Kwon and Thouless[5] indicate that the MSG phase also exists for a range of λ when $T=0$ ($p=1$). The MSG phase is similar to the phase which is observed below the de Almeida–Thouless line in the presence of a magnetic field, which will be discussed in Section 3. In this case it is the bond asymmetry, rather than an external field, which drives the transition.

In order to calculate χ_{EA} on the Bethe lattice, we consider two copies of the same realization of the lattice which are ferromagnetically coupled with bonds of strength R, as shown in Fig. 3. When $R=0$, the lattices decouple, and the fixed-point solution ρ for each of the lattices is simply the solution we have obtained in the companion paper. As we will show, χ^{EA} may be obtained directly from the moments of the corresponding fixed-point solution for the coupled lattice system, in the limit of the coupling

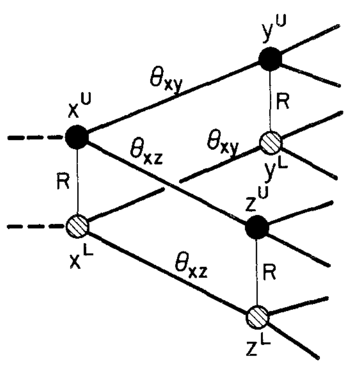

Fig. 3. The coupled copy lattice. Two copies of the same quench of the Bethe lattice are ferromagnetically coupled at corresponding sites σ_i^L on the lower lattice and σ_i^U on the upper lattice, with bonds of strength R.

strength $R \to 0$. Here we will calculate χ_{EA} in the neighborhood of the multicritical point using self-consistent estimates for these moments. To obtain the complete solution for the coupled lattices requires a rather complex bifurcation analysis, which we expect will produce the same results.

In zero external field the Hamiltonian for the coupled lattices is given by

$$H = \sum_{\langle i,j \rangle} J_{i,j}(\sigma_i^L \sigma_j^L + \sigma_i^U \sigma_j^U) - R \sum_i \sigma_i^L \sigma_i^U \tag{2}$$

where the bonds are of equal strength but random sign, $J_{i,j} = J\theta_{i,j}$, with the $\theta_{i,j}$ quenched and distributed independently according to (3), and where σ_i^L is the ith spin on the lower lattice, which is ferromagnetically coupled to σ_i^U on the upper lattice. For the reader who is unfamiliar with the benefits of working with coupled lattices, we include the following digression. The partition function is written $Z = \mathrm{Tr}\, e^{-\beta H}$, where $\beta = 1/k_B T$. From this we calculate the Helmholtz free energy: $A = -(1/\beta)\, \overline{\ln Z}$. Differentiating A once with respect to R, we obtain

$$\frac{dA}{dR} = \frac{1}{N} \overline{\sum_i \langle \sigma_i^U \sigma_i^L \rangle} \tag{3}$$

which at $R = 0$ is equal to q_{EA} provided that

$$\langle \sigma_i^U \rangle = \langle \sigma_i^L \rangle \tag{4}$$

for all i. A second differentiation with respect to R yields χ_{EA} at $R = 0$:

$$\chi_{EA} = \left. \frac{d^2 A}{dR^2} \right|_{R=0} = \frac{1}{N^2} \overline{\sum_{i,j} \langle \sigma_i^U \sigma_j^L \rangle^2 - \langle \sigma_i^U \rangle^2 \langle \sigma_j^L \rangle^2} \tag{5}$$

For the half-space lattice, χ_{EA} is given by

$$\chi_{EA} = \frac{1}{N} \overline{\sum_i \langle \sigma_x^U \sigma_i^L \rangle^2 - \langle \sigma_x^U \rangle^2 \langle \sigma_i^L \rangle^2} \tag{6}$$

where σ_x is the spin at the origin. Thus, the coupled lattices together with the restriction (4) provide a straightforward method of calculating the Edwards–Anderson order parameter and the Edwards–Anderson susceptibility.

The lattice formed from coupled Bethe lattices is clearly more complicated than the single lattice because it has loops. Nonetheless, if we consider each pair of spins σ_i^L, σ_i^U as a unit, we see that the treelike structure is maintained. Hence, we can still derive recursion relations for this system. We give only the results here. The derivation is given in Theorem A.3 of Appendix A of the companion paper.

Define the followinig random variables:

$$
\begin{aligned}
Q_x &= \langle \sigma_x^L \sigma_x^U \rangle \\
S_x &= \theta(\langle \sigma_x^L \rangle + \langle \sigma_x^U \rangle) \\
D_x &= \theta(\langle \sigma_x^L \rangle - \langle \sigma_x^U \rangle)
\end{aligned}
\tag{7}
$$

and the corresponding quantities for the neighboring sites y and z. In terms of these we obtain the following set of coupled recursion relations:

$$Q_x = \frac{G + rF}{F + rG} \tag{8}$$

$$S_x = \frac{\theta p(1+r)[S_y + S_z + p^2(S_y Q_z + S_z Q_y)]}{F + rG} \tag{9}$$

$$D_x = \frac{\theta p(1-r)[D_y + D_z - p^2(D_y Q_z + D_z Q_y)]}{F + rG} \tag{10}$$

where F and G are given by

$$F = 1 + \tfrac{1}{2} p^2 S_y S_z + p^4 Q_y Q_z \tag{11}$$

$$G = p^2(Q_y + Q_z + \tfrac{1}{2} S_y S_z) \tag{12}$$

and $p = \tanh(\beta J)$ and $r = \tanh(\beta R)$. Because of the restriction (4) mentioned above, in order to study χ_{EA} of the single lattice, we set $D_x = 0$ for our calculations. Consequently, when $r = 0$, S_x is just twice the magnetization X of the uncoupled lattice.

Now Q_x is simply the summand which appears in Eq. (3). Therefore, for the half-space lattice χ_{EA} is given by

$$\chi_{\mathrm{EA}} = E\left(\frac{dQ_x}{dR}\right) = \operatorname{sech}^2(\beta R)\, E\left(\frac{dQ_x}{dr}\right) \tag{13}$$

where E denotes the expectation with respect to the random bonds, as usual.

We calculate χ_{EA} using self-consistent estimates for the moments of the fixed-point densities of Q and S. The calculation is similar to our previous moment analysis; however, because we assume the existence of a solution for these densities, we present the calculation of the phase boundary as a conjecture. The ambitious reader could verify the existence of this solution by proving the stability of our previous solution to perturbations in r, and then using the implicit function theorem to extend the old solution to $r > 0$. Although we have not explicitly carried out all of the details, we anticipate no difficulty with this procedure, since as $r \to 0$ in the ferromagnetic phase, estimates of the moments of the density for S reduce to known moments of the old solution.

Conjecture 1. Define $\Delta = p - p_G$ and $\zeta = \Delta^{-1}(\lambda - \lambda^N)$. In the neighborhood of the multicritical point, the phase boundary between the magnetized spin-glass and the ferromagnetic phases is given asymptotically by

$$|\lambda - \lambda_N|^{1/2} = (3p_G)^{1/2}\,|p - p_G| \tag{14}$$

Remark. Bacause the coefficient of $|p - p_G|$ is smaller in (92) of ref. 1 than it is in (14) here, there is nonzero overlap of glassy and magnetized phases, which implies the existence of the intermediate MSG phase in the neighborhood of the multicritical point.

Calculation. The phase boundary is determined by divergence of χ_{EA}. We calculate χ_{EA} using self-consistent estimates for the moments. Let $r \ll \Delta$. We assume $Q \equiv E(Q_x) = O(\Delta)$, $M \equiv E(S_x) = O(\Delta)$, and $\Sigma \equiv E(S_x^2) = O(\Delta)$, which, when $r = 0$, corresponds to the correct estimates for the magnetized solution obtained in Section 6.2 of the companion paper. Using Eqs. (8) and (9) and keeping terms to $O(\Delta^3)$ and $O(r)$, we obtain the following three equations relating M, Q, and Σ:

$$0 = (2p^2 - 1)Q + \frac{p^2M^2}{2} - \frac{p^4\Sigma^2}{4} - 2p^6Q^3 + \frac{p^6Q\Sigma^2}{4} + \frac{p^6\Sigma^3}{4} + r + O(\Delta^4) \tag{15}$$

$$0 = -1 + 2p(2\lambda - 1)\left[1 + p^2Q - \frac{p^2\Sigma}{2} - p^4\Sigma Q + \frac{p^4\Sigma^2}{2} - p^4Q^2\right] + O(\Delta^4) \tag{16}$$

$$0 = (2p^2 - 1)\Sigma + 2p^2M^2 + 4p^2M^2 + 4p^2\Sigma Q - 2p^6\Sigma Q + \frac{11p^6\Sigma^3}{4} - 6p^6\Sigma^2Q + O(\Delta^4) \tag{17}$$

Combining these, we can in principle solve for Q, and differentiate with respect to r to obtain the phase boundary. In practice, it is easier to combine (15) and (17) to eliminate the M dependence. This results in a fourth equation,

$$0 = (2p^2 - 1)[Q - \tfrac{1}{4}\Sigma] - p^4\Sigma Q + \tfrac{1}{4}p^4\Sigma^2 + \tfrac{1}{2}p^6\Sigma Q^2 + \tfrac{7}{4}p^6\Sigma^2 Q - 2p^6\Sigma^2 Q - 2p^6Q^3 - \tfrac{7}{16}p^6\Sigma^3 + r + O(\Delta^4) \tag{18}$$

Next we differentiate Eqs. (16) and (18) with respect to Δ. From (18) we obtain

$$\Sigma' = Q'[2 - 4p^2\Sigma - 2p^2Q + O(\Delta^2)] \tag{19}$$

where the prime denotes the derivative with respect to r. Substituting this value of Σ' into the corresponding equation obtained from differentiating (16) results in

$$\begin{aligned} 0 = 1 + Q'[2p_G\Delta + \Delta^2 - \tfrac{1}{2}Q - 4p_G\,\Delta Q + \tfrac{11}{8}\Sigma Q - \tfrac{23}{64}\Sigma^2 \\ - \tfrac{3}{8}Q^2 + 2p_G\Delta\Sigma + p_G\Delta Q + O(\Delta^2)] \end{aligned} \tag{20}$$

We find that $\chi_{\mathrm{EA}} \propto Q'$ diverges when $\zeta = 3p_G\Delta + O(\Delta^2)$. To see this explicitly, we solve Eqs. (16) and (18) for the leading behavior of Σ and Q:

$$\Sigma = 16p_G\Delta + O(\Delta^2) \tag{21}$$

and

$$Q = 4p_G\Delta + 8p_G\zeta\Delta + O(\Delta^3) \tag{22}$$

Substituting these values into (20), we find

$$\chi_{\mathrm{EA}} = \frac{\operatorname{sech}^2(\beta R)}{6\Delta^2 - 4p_G\zeta\Delta + O(\Delta^3)} \tag{23}$$

Thus, to leading order, χ_{EA} diverges crossing the line $\zeta = 3p_G\Delta$, with critical exponent $\gamma = 1$, as desired. ▌

3. SPIN-GLASS TRANSITION IN AN EXTERNAL FIELD

In this section we show that on the Bethe lattice, the spin-glass transition persists in the presence of a small external field. We determine the asymptotic form of the de Almeida–Thouless (AT) line (see Fig. 2) by calculating the Edwards–Anderson susceptibility χ_{EA} [Eq. (6)], which diverges at the transition.

For simplicity in this analysis, we use a symmetric distribution of bonds and a symmetric distribution of external fields. As in the previous section, we perform our calculations on a system of two ferromagnetically coupled copies of the lattice (Fig. 3). The Hamiltonian is given by

$$H_{\mathrm{EA}} = -\sum_{\langle i,j\rangle} J_{i,j}(\sigma_i^U\sigma_j^U + \sigma_i^L\sigma_j^L) - \sum_i H_i(\sigma_i^U + \sigma_i^L) - R\sum_i \sigma_i^U\sigma_i^L \tag{24}$$

where σ_i^U and σ_i^L denote the spins at site i on the upper and lower lattices, respectively, the bonds $J_{i,j}$ are of equal strength, $J_{i,j} = J\theta_{i,j}$, and are independently and identically distributed according to

$$\theta_{i,j} = \begin{cases} +1, & \text{with prob. } 1/2 \\ -1, & \text{with prob. } 1/2 \end{cases} \tag{25}$$

The external fields are of equal strength $H_i = H\phi_i$, and are distributed independently according to

$$\phi_i = \begin{cases} +1, & \text{with prob. } 1/2 \\ -1, & \text{with prob. } 1/2 \end{cases} \tag{26}$$

and the lattices are coupled ferromagnetically with bonds of strength R.

The recursion relations for the coupled lattice system are derived in Appendix A of the companion paper, Theorem A.3. We give only the results here. As in Section 2, at each site we define the following quantities:

$$\begin{aligned} Q_x &= \langle \sigma_x^U \sigma_x^L \rangle \\ S_x &= \langle \sigma_x^U \rangle + \langle \sigma_x^L \rangle \\ D_x &= \langle \sigma_x^U \rangle - \langle \sigma_x^L \rangle \end{aligned} \tag{27}$$

In terms of these quantities we have the following set of coupled recursion relations:

$$Q_x = \frac{G_{yz} + rF_{yz} + h^2(F_{yz} + rG_{yz}) + h\phi_x(1+r)N_{yz}^s}{F_{yz} + rG_{yz} + h^2(G_{yz} + rF_{yz}) + h\phi_x(1+r)N_{yz}^s} \tag{28}$$

$$S_x = \frac{(1+r)[(1+h^2)N_{yz}^s + 2h\phi_x E_{yz}]}{F_{yz} + rG_{yz} + h^2(G_{yz} + rF_{yz}) + h\phi_x(1+r)N_{yz}^s} \tag{29}$$

$$D_x = \frac{(1-r)(1-h^2)N_{yz}^d}{F_{yz} + rG_{yz} + h^2(G_{yz} + rF_{yz}) + h\phi_x(1+r)N_{yz}^s} \tag{30}$$

where

$$G_{yz} = p^2[Q_y + Q_z + \tfrac{1}{2}\theta_y\theta_z(s_y s_z - d_y d_z)] \tag{31}$$

$$F_{yz} = 1 + \tfrac{1}{2}p^2\theta_y\theta_z(s_y s_z + d_y d_z) + p^4 Q_y Q_z \tag{32}$$

$$N_{yz}^s = p[\theta_y s_y(1 + p^2 Q_z) + \theta_z s_z(1 + p^2 Q_y)] \tag{33}$$

$$N_{yz}^d = p[\theta_y d_y(1 - p^2 Q_z) + \theta_z d_z(1 - p^2 Q_y)] \tag{34}$$

$$E_{yz} = p^2\theta_y\theta_z s_y s_z + (1 + p^2 Q_y)(1 + p^2 Q_z) \tag{35}$$

and where $r = \tanh \beta R$, $p = \tanh \beta J$, and $h = \tanh \beta H$, and y and z are the two sites connected to the origin x.

As shown in the previous section, χ_{EA} is proportional to the derivative of $Q = E(Q_x)$ with respect to r, evaluated at $D_x = 0$ and $r = 0$ [Eq. (13)]. We determine χ_{EA} by evaluating the moments of the fixed point of the coupled lattice system, which, at $h = 0$ and $r = 0$, corresponds to the symmetric spin-glass solution determined in Section 4.3 of ref. 1. Because the bond and field distributions are symmetric, there is no support for an asymmetric fixed point. It we begin with an asymmetric distribution of spins on the boundary, a single iteration of the recursion relations (28)–(30) results in a symmetric distribution on the first level. This allows us to ignore odd terms in the moment expansions, which greatly simplifies the analysis. We expect that a similar analysis for an asymmetric distribution of bonds (provided that one remains in the spin-glass phase, i.e., $\lambda < \lambda_N$) or fields will show that the spin-glass transition will also persist in an external field under these circumstances.

Again, since we are assuming the existence of a solution to the coupled system, we present the results as a calculation. To verify existence, it would suffice to establish the stability of the symmetric solution of ref. 6 to perturbations in r and h and then extend the solution using the implicit function theorem. As before, we expect no difficulties with this, since at $r = 0$ and $h = 0$ the moments of the system (28)–(30) reduce to those of the old solution.

Conjecture 2. Let the Hamiltonian be as specified in Eq. (24). In the presence of small external fields the spin-glass transition persists, and is characterized by diverging χ_{EA}. The asymptotic form of the de Almeida–Thouless line is given by

$$h^2 = 32 p_G \, |p - p_G|^3 \tag{36}$$

where $p = \tanh \beta J$ and $h = \tanh \beta H$. The critical exponent for the susceptibility is $\gamma = 1$.

Calculation. Let $\Delta = p - p_G$, and let $h \ll \Delta$ and $r \ll \Delta$. We obtain χ_{EA} using self-consistent estimates of the moments of the fixed-point distribution for the coupled lattice system. To that end, we define $Q = E(Q_x)$ and $\Sigma = E(S_x^2)$, and assume that $Q = O(\Delta)$, $\Sigma = O(\Delta)$, and $M = E(S_x) = 0$. Using Eqs. (28) and (29) and keeping terms of $O(\Delta^3)$, $O(h^2)$, and $O(r)$, and neglecting odd moments and terms of $O(r\Delta)$, $O(h^2\Delta)$, and $O(\Delta^4)$, we find that

$$Q = 2p^2 Q - \tfrac{1}{4} p^4 \Sigma^2 + \tfrac{1}{8} p^6 \Sigma t - 2p^6 Q T + h^2 + r \tag{37}$$

and

$$\Sigma = 2p^2\Sigma + 4p^4Q\Sigma - 2p^4\Sigma^2 + \tfrac{3}{2}p^6\Sigma t - 4p^6\Sigma L + 2p^6\Sigma T - 4p^6QL + h^2 \quad (38)$$

where $t = E(S_x^4)$, $T = E(Q_x^2)$, and $L = E(Q_x S_x^2)$. Similarly, we obtain t, T, and L to lowest order:

$$r \approx 3\Sigma^2 \quad (39)$$

$$T \approx Q^2 + \tfrac{1}{8}\Sigma^2 \quad (40)$$

$$L \approx Q\Sigma + \tfrac{1}{2}\Sigma^2 \quad (41)$$

In order to simplify our notation, in (37) and (38) as well as what follows, equals ($=$) will mean equality modulo terms of $o(r\Delta)$, $o(h^2\Delta)$, and $o(\Delta^4)$; the notation $\approx$ [as in (39)–(41)] will indicate equality to lowest order. Substituting (39)–(41) into Eqs. (37) and (38) yields a pair of equations relating Q and Σ:

$$Q = 2p^2Q - \frac{1}{4}p^4\Sigma^2 + \frac{p^6}{4}(\Sigma^3 + Q\Sigma - 8Q^3) + h^2 + r \quad (42)$$

and

$$\Sigma = 2p^2\Sigma + 4p^4Q\Sigma - 2p^4\Sigma^2 + p^6(\tfrac{11}{4}\Sigma^3 - 6\Sigma^2Q - 2\Sigma Q^2) + h^2 \quad (43)$$

To solve these, we define $\delta = Q - \frac{1}{4}\Sigma$. Subtracting one-fourth of Eq. (43) from Eq. (42), we obtain

$$\delta = 2p^2\delta - p^{\delta}\Sigma + \tfrac{7}{4}p^6\delta\Sigma^2 - 2p^6\delta Q^2 + r \quad (44)$$

Solving for δ yields

$$\delta = \frac{r}{(1/4)\Sigma - q_0 - (13/32)\,p^2\Sigma^2 + O(\delta\Sigma)} \quad (45)$$

where $q_0 = (2p^2 - 1)/4p^4$. From Eq. (42) we find that

$$\frac{1}{4}\Sigma - q_0 = \delta + \frac{4\hbar^2}{\Sigma} + \frac{9}{32}p^2\Sigma^2 + O(\delta\Sigma) \quad (46)$$

Note that this implies

$$\begin{aligned} \tfrac{1}{4}\Sigma &= q_h + \delta \\ Q &= q_h + 2\delta \end{aligned} \quad (47)$$

where

$$q_h = q_0 + \frac{9}{4} q_0^2 + \frac{h^2}{q_0} \tag{48}$$

If follows that $\chi_{EA} = 2\,\mathrm{sech}^2(\beta R)(d\delta/dr)|_{r=0}$. Substituting (46) into (45) yields

$$\delta = \frac{r}{\delta + A + O(\delta\Sigma)} \tag{49}$$

where

$$A = \frac{h^2}{q_0} - q_0^2 \tag{50}$$

We will show that $A = 0$ defines the de Almeida–Thouless line. Solving (49) for δ, we find that

$$\delta \approx \tfrac{1}{2}[-A \pm (A^2 + 4r)^{1/2}] \tag{51}$$

In Eq. (51), there are two criteria which could be used to choose the sign of the square root. The first is to require that χ_{EA} be positive, which always corresponds to choosing the positive root. The second is to choose the root which (when $h = 0$) corresponds to our symmetric spin-glass solution for the single lattice in the limit as $r \to 0$. When $A > 0$ (i.e., above the AT line), we find that both criteria require that we choose the positive root, and that asymptotically

$$\delta \approx \frac{r}{A} \tag{52}$$

Consequently,

$$\chi_{EA} = \frac{2\,\mathrm{sech}^2(\beta R)}{A} \tag{53}$$

In other words, χ_{EA} diverges with exponent $\gamma = 1$ as we approach the AT line ($A = 0$) from above. The asymptotic form of the AT line is given by

$$h^2 = q_0^3 = 32 p_G \Delta^3 \tag{54}$$

as specified in the statement of the proposition. ∎

Remark. Interestingly, when $A < 0$ (i.e., below the AT line), we find that the two criteria correspond to choosing opposite roots in (51). For an asymmetric distribution of bonds in zero field, this phenomenon is also encountered in the spin-glass and magnetized spin-glass phases. This is currently a topic of great interest and discussion.

4. CONCLUSIONS AND COMPARISON WITH THE INFINITE-RANGE MODEL

In this and the companion paper, we have presented an analysis of the static properties of the Bethe lattice spin glass with fixed uncorrelated boundary conditions. Mathematical rigor requires some care; however, all of our results follow from arguments which are relatively simple in principle. The treelike structure of the lattice allows us to derive recursion relations (Appendix A in ref. 1) giving properties [e.g., magnetization in (11) of ref. 1] of a given site in terms of the same properties of the sites connected to it. Analysis of the recursion relations allows us to determine the phase diagram near the high-temperature paramagnetic phase. The paramagnetic phase boundaries for the spin-glass and ferromagnetic phases follow from a simple linear stability analysis of the paramagnetic solution $\rho(X)=\delta(X)$ (ref. 1, Section 3). At the ferromagnetic transition, $\rho(X)$ develops an instability with respect to small perturbations to the mean of $\rho(X)$; at the spin-glass transition, $\rho(X)$ becomes unstable with respect to perturbations in the width. Similarly, the phase boundary separating the spin-glass and magnetized spin-glass phases is associated with an instability in the symmetric spin-glass solution with respect to perturbations in the mean of the solution (ref. 1, Section 6.1). Finally, the divergence of the Edwards–Anderson susceptibility determines the phase boundary between the ferromagnet and the magnetized spin glass (this paper, Section 2), as well as the de Almeida–Thouless line (Section 3).

A dynamical systems analysis of the recursion relation allows us to determine the density of single-site magnetization $\rho(X)$, which characterizes the bulk order in the phase. The density $\rho(X)$ is the attractor of the dynamical system, and this analysis shows explicitly how statistical mechanical phase transitions in this model correspond to bifurcations in a dynamical system. In the paramagnetic phase, the unique globally attracting solution of the recursion relation is $\rho(X)=\delta(X)$. At the spin-glass transition, the paramagnetic solution becomes unstable, and a symmetric solution emerges (ref. 1, Sections 4.2 and 4.3). To leading order in $p-p_G$, the spin-glass solution is a Gaussian, and the higher-order corrections are given in terms of Gaussians multiplied by Hermite polynomials. Analysis of the ferromagnetic transition leads us to discover a continuous family of complete analytic functions, and the ferromagnetic solution is given in terms of these (ref. 1, Section 5).

At this point it is useful to compare our results with the corresponding results for the infinite-range model. The Hamiltonian for the infinite-range model is given by

$$H = -\sum_{(i,j)} J_{i,j}\sigma_i\sigma_j \tag{55}$$

where the bonds $J_{i,j}$ are typically chosen from a Gaussian distribution with width $J/\sqrt{N}$ and mean $J_0/\sqrt{N}$, and the sum is over all pairs of spins.

Using the replica trick, Sherrington and Kirkpatrick proposed the first solution for this model.[3] Their solution is referred to as the replica-symmetric solution, because it is invariant with respect to permutations of N identical replicas $(\alpha, \beta,...)$ of the system. For example,

$$q_{\alpha\beta} = \overline{\langle\sigma_i^\alpha\rangle\langle\sigma_i^\beta\rangle} = q_{\text{EA}}, \qquad \forall\alpha, \beta \tag{56}$$

However, in the spin-glass phase the replica-symmetric solution is incorrect. One manifestation of this fact is that the entropy of this solution is seen to be negative at low temperatures. In addition, the EA susceptibility is negative and the solution is unstable.[2]

The presumably correct solution to the SK model was obtained by Parisi, and is referred to as the replica-symmetry-breaking solution.[4] In Parisi's solultion, quantities involving a single replica are equal (e.g., $m_\alpha = m_\beta$). However, his solution breaks replica permutation symmetry. The Parisi solution begins with an ansatz for a hierarchy of overlaps $q_{\alpha\beta}$, which leads to an infinite number of extremal states. The Parisi solution has good agreement with simulations of the SK model,[7] and is stable.

To make a comparison between our solution on the Bethe lattice and these solutions, we considered the generalization of our results to a lattice with forward branching ratio K, and looked at the limit as $K \to \infty$. The recursion relation for a Bethe lattice with forward branching ratio K is given by

$$X =_d \frac{\prod_{i=1}^K (1 + p\theta_i Y_i) - \prod_{i=1}^K (1 - p\theta_i Y_i)}{\prod_{i=1}^K (1 + p\theta_i Y_i) + \prod_{i=1}^K (1 - p\theta_i Y_i)} \tag{57}$$

where each site y_i is connected to the origin x by a bond of sign θ_i, and $X = \langle\sigma_x\rangle$ with Y_i defined accordingly. As $K \to \infty$, this reduces to the familiar relation describing the replica-symmetric solution of the SK model:

$$X =_d \tanh(\alpha q^{1/2} Z) \tag{58}$$

where $\alpha = p/\sqrt{K}$, q is the second moment of the replica-symmetric solution $\rho(X)$, and Z is a Gaussian of unit width. In addition, working directly with

(57), we have verified that as $K \to \infty$, the second moment q_K of the distribution $\rho_K(X)$ converges to q to second order in $\alpha - \alpha_c$. These results are also similar to results obtained for the replica-symmetric solution to the dilute long-range model.[8,9]

The above results lead us to believe that in a certain sense our solution corresponds to the replica-symmetric solution of the SK model. A natural question to ask is whether replica symmetry breaking could be associated with long-range correlations in the boundary conditions,[10] which has the effect of introducing loops into the system. To examine this possibility more carefully, recently Chayes and Chayes looked at the stability of our solution with respect to various types of long-range correlations in the boundary conditions.[11] For example, they examined the stability of our solution with respect to correlations which decay with the distance between spins, and, alternatively, they coupled all spins on the boundary equally. In every physically reasonable case, they found that our solution was stable with respect to the imposed correlations. This suggests that, given a chance, the Bethe lattice will choose the replica-symmetric solution.

Finally, it is worth noting that a solution which may correspond to the replica-symmetry-breaking solution does exist on the coupled lattice system studied in Section 2 and 3. In particular, as suggested in the remark at the end of Section 3, in the spin-glass phase (i.e., when $A < 0$), there appears to be a solution with positive susceptibility which is distinct from the replica-symmetric solution. Further study of the coupled system may lead to a better understanding of the relationship between the Bethe lattice and the infinite-range model.

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